Nuclear Structure from Gamma-Ray Spectroscopy

2019 Postgraduate Lectures

Lecture 6: Cranking

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Cranking

A rotation is externally imposed on a nucleus about the x axis. The Schrödinger equation is:

$$i\hbar \ \partial \Psi_{lab} / \partial t = H_{lab} \Psi_{lab}$$

Using the rotation operator

with $\Psi_{lab} = R_x \Psi_{int}$ and $H_{lab} = R_x H_{int} R_x^{-1}$, the SE within the intrinsic frame becomes:

iħ
$$\partial \Psi_{int} / \partial t = [H_{int} - \hbar \omega I_x] \Psi_{int}$$

The Routhian

 The cranked Hamiltonian or <u>Routhian</u> (just the energy in the rotating frame) is then:

$$H^{\omega} = H_{int} - \hbar \omega I_{x}$$

where $\textbf{wI}_{\textbf{x}}$ is analogous to the classical Coriolis and centrifugal forces

 In terms of single-particle states, the cranking Hamiltonian is:

$$H^{\omega} = \sum_{i} h^{\omega}(i) = \sum_{i} [h_{int}(i) - \hbar \omega j_{x}(i)]$$

where $j_x(i)$ are the components of the nucleonic angular momenta on the rotation axis (x)

Single-Particle Routhians

 The single-particle <u>Routhian</u> can be evaluated by solving the eigenvalue equation:

$$h^{\omega}|v^{\omega}\rangle = e_{v}^{\omega}|v^{\omega}\rangle$$

where $|\mathbf{v}^{\boldsymbol{w}}\rangle$ are the single-particle eigenfunctions in the rotating frame

• The Routhian is simply the energy in the rotating frame of reference: e_v^{ω}

The Alignment

• The alignment is just the expectation value of j_x : $\langle j_x \rangle$ and is equal to the (negative) differential of the Routhian with respect to rotational frequency, i.e.

 $de_v^{\omega}/d\omega = -\hbar \langle v^{\omega} | j_x | v^{\omega} \rangle$

- Those orbits with large $j_{\rm x}$ values, and hence low Ω values, are most affected by the rotation, i.e. the Coriolis and centrifugal forces

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Symmetries of Rotating Nuclei

- The time-reversal (twofold) degeneracy of the $\pm \Omega$ states is lifted by the wj_x term
- Axially symmetric potentials exhibit invariance with respect to rotations by $180^{\circ}(\pi)$ about the three principal axes (reflection symmetry)
- However, the cranking Hamiltonian is only invariant for rotation of π about the x axis

$$R_{x}(\pi) = exp(-i\pi I_{x})$$

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Signature Quantum Number

• A rotation of 2π leaves the wavefunction unchanged, except for a possible phase factor (±1), i.e.

 $R_{x}^{2}(\pi) \Psi = r^{2} \Psi = (-1)^{A} \Psi$ with $r^{2} = \pm 1$

- The eigenvalues r of the rotation operator $R_x(\pi)$, called signature, are good quantum numbers, i.e. constants of the motion
- The signature exponent quantum number α is defined:

 $r = exp(-i\pi\alpha)$

Signature Quantum Numbers



• The signature is:

$$r = \pm 1 \text{ (even A)} \quad \alpha = 0, 1$$

$$r = \pm i \text{ (odd A)} \quad \alpha = \pm \frac{1}{2}$$

The spins are restricted according to

$$\alpha = I \mod 2$$

Only good quantum numbers: π and α

Total signature:

$$\alpha_{tot}$$
 = $\sum \alpha$

Total parity:



Signature Partners



- Signature is a good quantum number at high spin
- A splitting between the
 α = 0/1 (even A) or α =
 ±¹/₂ (odd A) states gives
 rise to two distinct
 'signature partner' bands

• For an orbital with angular momentum j the 'favoured' band has signature: $\alpha_f = j \mod 2$

CSM Calculations

Including pairing the CSM Hamiltonian is:

$$H_v^{\omega} = H_{sp} - \Delta(P^{\dagger} + P) - \Lambda \tilde{N} - \omega I_x$$

with $H^{\omega} = \sum h^{\omega}$, $H_{sp} = \sum h_{sp}$, $I_x = \sum j_x$ and the total energy $E = \sum e_v^{\omega}$, i.e. a sum of the single-particle Routhians, $e_v^{\omega} = \langle h_v^{\omega} \rangle$

• The parameters λ (- $\partial E/\partial N$) and ω (- $\partial E/\partial I$) can be considered as <u>Lagrange multipliers</u> needed to constrain the particle number and angular momentum, respectively: $\langle \tilde{N} \rangle = N$ and $\langle I_x \rangle = \int [I(I+1) - K^2]$

Quasiparticle Diagrams



Here is an example of a Woods-Saxon quasiparticle diagram

 $\begin{array}{c} \underline{\text{Label}} & (\pi, \alpha) \\ A (C) & (+, +\frac{1}{2}) \\ B (D) & (+, -\frac{1}{2}) \\ E (G) & (-, -\frac{1}{2}) \\ F (H) & (-, +\frac{1}{2}) \end{array}$

Comparison to Experiment



- The results of cranking calculations provide Routhians rather than energies
- The experimental data must be transformed into the intrinsic frame to afford detailed comparisons
- Quantities are approximated as quotients of finite differences

Moments of Inertia

The frequency for y ray 1 is:

 $\omega(I) = (1/\hbar) dE(I)/dI_x \approx E_{\gamma 1}/2\hbar$

Expressions for the moments of inertia are:

$$\Im^{(1)}(I) = \hbar^{2} \{ (2I+1) / (E(I+1) - E(I-1)) \} = \frac{\hbar^{2} (2I+1) / E_{\gamma 1}}{\Im^{(2)}(I-1)} = \frac{\hbar^{2} \{ 4 / (E_{\gamma 1} - E_{\gamma 2}) \}}{4\hbar^{2} / \Delta E_{\gamma}}$$

 The dynamic moment of inertia corresponds to spin I-1 and the associated frequency should be calculated at I-1. In practice, the average frequency is used:

$$\omega$$
(I-1) \approx (E_{y1} + E_{y2}) / 4ħ

Experimental Routhian and Alignment

The experimental Routhian may be expressed as:

$$E_{expt}^{\omega}(I) = \frac{1}{2} \{ E(I+1) + E(I-1) \} - \omega(I) I_{x}(I) \}$$

but we are interested in obtaining a quasiparticle Routhian and so must subtract the collective rotational energy:

$$e'(I) = E^{\omega}_{expt}(I) - E^{\omega}_{ref}(I)$$

Similarly, we remove the collective spin to produce the quasiparticle alignment:

$$i_x(I) = I_x(I) - I_{x,ref}(I)$$

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Choice of Reference

- The reference removes the collective effects of rotation and leaves the energies and spins solely from the valence quasiparticles, in the rotating frame
- The reference can be obtained from the ground state band (zero quasiparticle, vacuum) of a (neighbouring) even-even nucleus
- At low spin, it is found that ℑ ∝ w². Hence a 'variable moment of inertia' (VMI) reference can be fitted, introducing '<u>Harris Parameters</u>' ℑ₀, ℑ₁:

$$\mathfrak{I}^{(1)}_{ref} = \mathfrak{I}_0 + \mathfrak{I}_1 \omega^2$$

Harris Parameters

The Harris Parameters can be obtained by fitting:

$$\mathbf{I}_{\mathsf{x},\mathsf{ref}}(\boldsymbol{\omega}) = \boldsymbol{\omega}\{\mathfrak{I}_0 + \mathfrak{I}_1\boldsymbol{\omega}^2\} + \mathsf{i}_{\mathsf{x}}$$

to the reference band. Note that $i_x = 0$ for the groundstate band of an even-even nucleus

• The energy reference is then given by:

$$\mathsf{E}^{\mathsf{w}}_{\mathsf{ref}} = -\hbar \int \mathbf{I}_{\mathsf{x},\mathsf{ref}} \mathsf{d} \mathsf{w} = -\frac{1}{2} \mathsf{w}^2 \,\mathfrak{I}_0 - \frac{1}{4} \, \mathsf{w}^4 \,\mathfrak{I}_1 + \frac{1}{8} \hbar^2 / \mathfrak{I}_0$$

where the final term, an integration constant, ensures that the ground-state energy, $E_{ref}^{w}(I=0)$, is zero

Expt. Alignments and Routhians



- Experimental results for N = 74 isotones are shown here
- A clear 'backbend' is seen for ¹³²Ce while the heavier nuclei show 'upbends'
- E and F correspond to proton h_{11/2} orbitals, and these are the quasiparticles that align

Band Crossings

- Band crossings can be classified by the rotational frequency at which they occur w_c and the gain in alignment Δi_x at the crossing
- Experimentally: These quantities can be obtained by plotting e' vs. w and i_x vs. w
- <u>Theoretically</u>: Crossing frequencies can be obtained from CSM quasiparticle diagrams.
- The gain is alignment is given by the slopes of the interacting levels, E and F:

$$\Delta i_x = -(1/\hbar) \{ de'_E/dw + de'_F/dw \}$$

Signature Splitting

The <u>signature splitting</u> between the components of an orbital is the difference in excitation energy (or Routhian) at a fixed frequency, e.g.

$$\Delta e'_{FE}(\omega) = e'_{F}(\omega) - e'_{E}(\omega)$$

- The magnitude of the signature splitting is related to the admixture of the $\Omega = \frac{1}{2}$ component in the wavefunction and is <u>larger</u> for <u>low</u> Ω values
- The Coriolis interaction connects states with $\Omega = \pm 1$

Staggering Parameter



 One way to enhance signature effects is to plot the

staggering parameter S(I)



Signature Inversion



- In some odd-odd nuclei at low spin the signatures are the 'wrong way round' i.e. the 'favoured' signature is energetically unfavoured !
- At higher spin the signatures revert to their 'expected' ordering

Strutinsky Shell Correction

- Shell correction energy
- Cranked Nilsson Strutinsky Model
- Total Routhian Surfaces

Shell Effects



- A nuclear 'property' (e.g. binding energy) usually shows an irregular behaviour with mass A (or Z or N). It is made up of an oscillatory part ΔE on top of a smooth part E_{smooth}
- Strutinsky's idea was to use the shell model to obtain ΔE as the local variation from the average smoothed (shell model) value, but then to use the Liquid Drop Model to calculate the real 'smooth' behaviour E_{smooth}

Strutinsky Shell Correction



(a) Liquid drop: $g_F(e) = g_{AV}(e)$

(b) and (c) show shell effects.

A change in nuclear binding arises from: $g_{AV}(e) - g_{F}(e)$ To obtain both the global (liquid drop) and local (shell model) variations with δ, Z and A, Strutinsky developed a method to combine the best properties of both models

He considered the behaviour of the level density g(e) in the two models and calculated the 'fluctuation' energy

Level Density

Single-particle level density and shell effects



Strutinsky Procedure

• The nuclear (binding) energy is considered to have to have an oscillatory part ΔE_{shell} , caused by quantal effects (shell model), superposed upon a smoothly varying liquid drop part E_{LD} :

$$\mathsf{E} = \mathsf{E}_{\mathsf{LD}} + \Delta \mathsf{E}_{\mathsf{shell}}$$

• Strutinsky proposed that only ΔE_{shell} ('microscopic') should be calculated within the framework of the shell model, while the smoothly varying part E_{LD} ('macroscopic') should be taken from the Liquid Drop Model

Strutinsky Procedure (cont)

 Similarly, the total shell-model energy E_{SH} does not vary smoothly and is composed of oscillatory and smooth parts:

$$E_{SH} = \sum_{1}^{A} \epsilon_{i} = \breve{E}_{SH} + \Delta E_{SH}$$

- The <u>real</u> and <u>smoothed</u> level densities can be defined by g(c) and ğ(c), respectively
- The number of levels between ϵ and $\epsilon + d\epsilon$ is given by $g(\epsilon) d\epsilon$ and the level density is:

$$g(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$$

where $\delta(\epsilon - \epsilon_i)$ is the Dirac delta function

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Strutinsky Procedure (cont)

The particle number can be evaluated as:

The total shell-model energy and the smoothed part are then given, respectively, as:

$$E_{SH} = \int^{\Lambda} \epsilon g(\epsilon) d\epsilon$$
 and $\breve{E}_{SH} = \int^{\Lambda} \epsilon \breve{g}(\epsilon) d\epsilon$

• Note that $\Lambda \neq \Lambda'$ because of a smearing of the Fermi surface when calculating \breve{E}_{SH}

Strutinsky Procedure (cont)

 The total energy of the nucleus may finally be written as:

$$E = E_{LD} + \Delta E_{SH} = E_{LD} + [E_{SH} - \breve{E}_{SH}]$$

where E_{LD} is the macroscopic contribution and $[E_{SH} - \breve{E}_{SH}]$ is the microscopic shell correction

- Note that the shell correction can be positive or negative
- Negative values give increased binding and stability

Shell Correction Energies



Superheavy Island



 Shell effects, particularly hexadecapole deformation, can stabilise very heavy nuclei

 Such superheavy nuclei only exist because of subtle quantum mechanical effects leading to a localised region ('island') of increased stability

Strutinsky Method for Spin

- The Strutinsky technique can be extended to include rotation
- We introduce another 'level density':

$$g_2(\epsilon) = \sum_i \langle j_x \rangle_i \, \delta(\epsilon - \epsilon_i)$$

The total single-particle energy is obtained from the cranking Hamiltonian as:

$$\mathsf{E}_{\mathsf{SP}}(\mathsf{I}) = \int^{\Lambda} \varepsilon g(\varepsilon) d\varepsilon + \hbar \omega \mathsf{I}$$

where
$$I = \int^{A} g_2(\epsilon) d\epsilon$$

Strutinsky Method for Spin

The smoothed energy is:

$$\breve{E}(\epsilon) = \int^{\Lambda'} \epsilon \breve{g}(\epsilon) d\epsilon + \hbar \omega \breve{g}_2(\epsilon) d\epsilon$$

• The cranked Nilsson Strutinsky method includes deformation (ϵ_2 , ϵ_4 , γ) and spin

Cranked Nilsson Strutinsky

The total energy is:

$$\mathsf{E}\left(\varepsilon_{2},\varepsilon_{4},\gamma,\,\mathsf{I}\right)=\mathsf{E}_{\mathsf{LD}}\left(\varepsilon_{2},\varepsilon_{4},\gamma,\,\mathsf{I}\right)+\Delta\mathsf{E}_{\mathsf{SH}}\left(\varepsilon_{2},\varepsilon_{4},\gamma,\,\mathsf{I}\right)$$

The macroscopic energy contribution is can be calculated from:

$$E_{LD} = E_{surf} + E_{Coul} + (9\hbar I)^2 / 2\Im_{rig}$$

 This method usually ignores pairing correlations and is hence only valid for high-spin states (I > 20 ħ)

Total Routhian Surfaces

• This method is based on the Woods-Saxon potential and includes pairing. The total energy of a nucleus (Z, N) as a function of deformation $\beta' = (\beta_2, \beta_4, \gamma)$ is:

 $\mathsf{E}(\boldsymbol{\omega},\boldsymbol{Z},\mathsf{N},\boldsymbol{\beta}')=\mathsf{E}_{\mathsf{macro}}\left(\boldsymbol{\omega},\boldsymbol{Z},\mathsf{N},\boldsymbol{\beta}'\right)+\Delta\mathsf{E}_{\mathsf{shell}}\left(\boldsymbol{\omega},\boldsymbol{Z},\mathsf{N},\boldsymbol{\beta}'\right)$

+ $\Delta E_{pair}(\omega, Z, N, \beta')$

The total Routhian is:

 $\mathsf{E}(\omega, Z, \mathsf{N}, \beta') = \mathsf{E}(\omega = 0, Z, \mathsf{N}, \beta')$

+ $[\langle \Psi^{\omega} | H^{\omega}(Z,N,\beta') | \Psi^{\omega} \rangle - \langle \Psi | H^{\omega=0}(Z,N,\beta') | \Psi \rangle]$

- $\frac{1}{2}\omega^2[\Im_{macro}(A,\beta') - \Im_{Strut}(N,Z,\beta')]$

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Total Routhian Surfaces (cont)

- The term E(w=0,Z,N,β') corresponds to the liquid-drop energy, the single-particle shell correction energy, and the pairing energy at zero rotational frequency
- $[\langle \Psi^{w}|H^{w}(Z,N,\beta')|\Psi^{w}\rangle \langle \Psi|H^{w=0}(Z,N,\beta')|\Psi\rangle]'$ is the change in energy due to rotation
- The term $\frac{1}{2}\omega^2[\Im_{macro}(A,\beta') \Im_{Strut}(N,Z,\beta')]$ represents a renormalisation of the LDM energy which is required due to unrealistically large proton and neutron radii used in some parameterisations of the Woods-Saxon potential

TRS Maps

